Schwartz Inequality (Thm 5)

$$|\langle v|w\rangle| \leq |v||w|$$

meaning... recall vectors $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$|\cos\theta| \leq 1$

so $|\vec{A}\cdot\vec{B}| \leq |\vec{A}|\cdot|\vec{B}|$.

equality when $\cos\theta = \pm 1$

then $\vec{A} \parallel \vec{B}$ or $\vec{A} \downarrow\uparrow \vec{B}$

More generally...

consider... $|v\rangle + \lambda|w\rangle \equiv |z\rangle$

complex, arbitrary

must be: $\langle z|z\rangle \geq 0$

$$\{\langle v| + \lambda^*\langle w|\}\{|v\rangle + \lambda|w\rangle\} \geq 0$$

$$\langle v|v\rangle + |\lambda|^2\langle w|w\rangle + \lambda\langle v|w\rangle + \lambda^*\langle w|v\rangle \geq 0$$

- minimize left hand side
- complicated because $\lambda = \lambda_R + i\lambda_I$ (2 variables)
- $\langle v|w\rangle = \alpha + i\beta$

$$\lambda\langle v|w\rangle + \lambda^*\langle w|v\rangle = (\lambda_R + i\lambda_I)(\alpha + i\beta) + (\lambda_R - i\lambda_I)(\alpha - i\beta)$$

$$= 2\alpha\lambda_R - 2\beta\lambda_I$$

$$|\lambda|^2 = \lambda_R^2 + \lambda_I^2$$

left hand side:

$$f(\lambda) = \langle v|v\rangle + (\lambda_R^2 + \lambda_I^2)\langle w|w\rangle + 2\alpha\lambda_R - 2\beta\lambda_I$$

differentiate, set $= 0$

$$\frac{\partial f}{\partial \lambda_R} = 2\lambda_R\langle w|w\rangle + 2\alpha = 0 \qquad \frac{\partial f}{\partial \lambda_I} = 2\lambda_I\langle w|w\rangle - 2\beta = 0$$

$$\lambda_R = -\frac{\alpha}{\langle w|w\rangle} \qquad \lambda_I = \frac{\beta}{\langle w|w\rangle}$$

$$\lambda = \lambda_R + i\lambda_I = \frac{-\alpha + i\beta}{\langle w|w\rangle} = -\frac{\langle v|w\rangle^*}{\langle w|w\rangle} = -\frac{\langle w|v\rangle}{\langle w|w\rangle}$$

this λ is a minimum because

$$\frac{\partial^2 f}{\partial \lambda_R^2} = 2\langle w|w\rangle \geq 0 \qquad \frac{\partial^2 f}{\partial \lambda_I^2} = 2\langle w|w\rangle \geq 0$$

$$\frac{\partial^2 f}{\partial\lambda_R \partial\lambda_I} = \frac{\partial^2 f}{\partial\lambda_I\partial\lambda_R} = 0$$

so $$|z\rangle = |v\rangle - \frac{\langle w|v\rangle}{\langle w|w\rangle}|w\rangle$$

$$= |v\rangle - \frac{|w\rangle\langle w|v\rangle}{\langle w|w\rangle}$$

is <u>the</u> <u>shortest</u> $|z\rangle$ imaginable and it <u>still</u> must be greater than or equal to zero.

$$\langle z|z\rangle = \left(\langle v| - \frac{\langle v|w\rangle}{\langle w|w\rangle}\langle w|\right)\left(|v\rangle - \frac{\langle w|v\rangle}{\langle w|w\rangle}|w\rangle\right) \geq 0$$

$$\langle v|v\rangle - \frac{\langle v|w\rangle\langle w|v\rangle}{\langle w|w\rangle} - \frac{\langle w|v\rangle\cancel{\langle v|w\rangle}}{\cancel{\langle w|w\rangle}} + \frac{\langle v|w\rangle\cancel{\langle w|v\rangle}}{\cancel{\langle w|w\rangle}}\langle w|w\rangle \geq 0$$

$$\langle v|v\rangle - \frac{\langle v|w\rangle\langle w|v\rangle}{\langle w|w\rangle} \geq 0$$

$$|\langle v|w\rangle|^2 \leq |v|^2 |w|^2$$

$$|\langle v|w\rangle| \leq |v||w| \qquad \text{QED.}$$

Very Important for the uncertainty principle.

Subspaces

D11: Given $\mathbb{V}$, a set of elements that form a vector space among themselves is called a subspace. A particular subspace i of dimensionality n_i called $\mathbb{V}_i^{n_i}$

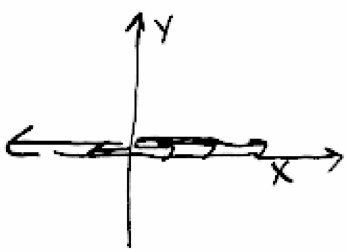
Examples: in 3-space (x,y,z) vectors with only x-components are a subspace. ($\mathbb{V}_x^1$) $\mathbb{V}^3(R)$
vectors with only x-y components (those in the x-y plane) form a subspace ($\mathbb{V}_{xy}^2$).

D11 about "breaking down" a vector space.
What about building up a vector space?

D12: Given two subspaces $\mathbb{V}_i^{n_i}$ and $\mathbb{V}_j^{m_j}$, $\mathbb{V}_i^{n_i} \oplus \mathbb{V}_j^{m_j}$, the sum, is defined as the set : (1) all elements of $\mathbb{V}_i^{n_i}$
(2) all elements of $\mathbb{V}_j^{m_j}$
(3) all possible linear combinations of elements of $\mathbb{V}_i^{n_i}$ and $\mathbb{V}_j^{m_j}$

Examples : $\mathbb{V}_x^1(R)$ and $\mathbb{V}_y^1(R)$ has all vectors of the form: $a\hat{x}$, $b\hat{y}$, $a\hat{x} + b\hat{y}$
#1 #2 #3

↑ don't forget



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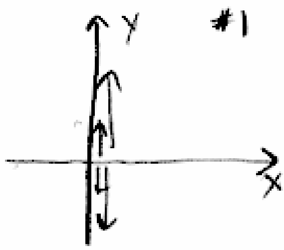

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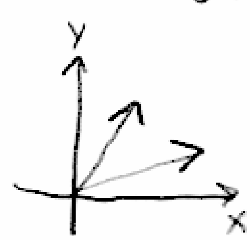

as you should expect, the dimensionality of $\mathbb{V}_i^{n_i} \oplus \mathbb{V}_j^{m_j}$ is, when all elements of $\mathbb{V}_i^{n_i}$ are orthogonal to all elements of $\mathbb{V}_j^{m_j}$, equal to $n_i + m_j$.

$$V_{xy}^2 \oplus V_{xz}^2 = \begin{matrix} 4 \\ \text{or } 3 \end{matrix} \; ?$$

not 4 because both vector spaces include x so 3 is the answer

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Addition of spaces is important!

Linear Operators

Operator: a list of instructions, to do to a ket (abstract vector) $|V\rangle$ and arrive at some <u>other</u> ket $|V'\rangle$

$\underset{\sim}{\Omega}$

Linear Operator: when the initial kets have a linear relationship, then the final kets have that same linear relationship.

Operator: $\underset{\sim}{\Omega}|V\rangle = |V'\rangle$

Linear Operator: when $|U\rangle = a|V\rangle$

$$\underset{\sim}{\Omega}|V\rangle = |V'\rangle$$

$$\underset{\sim}{\Omega}|U\rangle = |U'\rangle$$

then $|U'\rangle = a|V'\rangle$

aka $\underset{\sim}{\Omega}a|V\rangle = a\underset{\sim}{\Omega}|V\rangle$

more generally, when $|Z\rangle = \alpha|V\rangle + \beta|W\rangle$
then for a linear operator $\underset{\sim}{\Omega}|Z\rangle = \alpha\underset{\sim}{\Omega}|V\rangle + \beta\underset{\sim}{\Omega}|W\rangle$

Operators can also operate on the duals of vectors, that is, on bras as well as kets:

$$\langle W|\underset{\sim}{\Omega} = \langle W'|$$

since $\underset{\sim}{\Omega}$ a list of instructions, no problem

Linear operator:

$$(\alpha\langle W| + \beta\langle Z|)\underset{\sim}{\Omega} = \alpha\langle W|\underset{\sim}{\Omega} + \beta\langle Z|\underset{\sim}{\Omega}$$

<u>Examples</u>: $\underset{\sim}{\Omega} = \underset{\sim}{I}$ or $\underset{\sim}{\mathbb{1}}$ leave vector alone! linear

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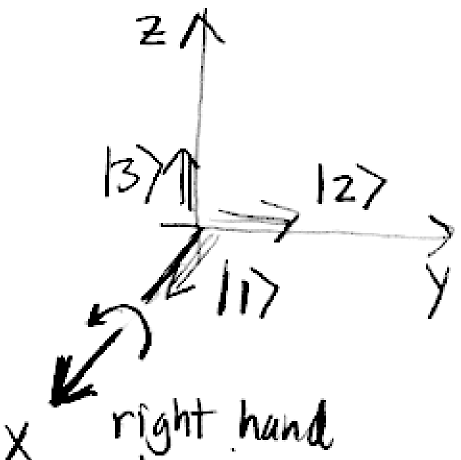
In xyz space $\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})$ read: rotate by $\tfrac{1}{2}\pi$ counterclockwise about $\hat{i}$

[General case: $\underset{\sim}{R}(\vec{\theta})$ means rotate counterclockwise by $|\vec{\theta}|$ about the $\vec{\theta}$ direction]



right hand rule is counterclockwise

$$\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})|1\rangle = |1\rangle$$

$$\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})|2\rangle = |3\rangle$$

$$\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})|3\rangle = -|2\rangle$$

Linear? From <u>figure</u> it is (not exactly obvious)!

Only need to evaluate action of operator on a basis... preferably an orthonormal basis.

evaluate: $\underset{\sim}{\Omega}|i\rangle = |i'\rangle$

then if $|V\rangle = \sum_{i=1}^{n} v_i |i\rangle$

$$\underset{\sim}{\Omega}|V\rangle = \sum_{i=1}^{n} v_i \underset{\sim}{\Omega}|i\rangle = \sum_{i=1}^{n} v_i |i'\rangle$$

That is, the components of $|V\rangle$ in the basis $|i\rangle$ will be the same as the components of $|V'\rangle = \underset{\sim}{\Omega}|V\rangle$ in the basis $|i'\rangle$, where $|i'\rangle = \underset{\sim}{\Omega}|i\rangle$

OPERATORS MAY NOT COMMUTE!

$\underset{\sim}{\Omega}, \underset{\sim}{\Lambda}$ are two operators (possibly linear)

$\underset{\sim}{\Lambda}(\underset{\sim}{\Omega}|V\rangle) = \underset{\sim}{\Lambda}\underset{\sim}{\Omega}|V\rangle$ means do $\underset{\sim}{\Omega}$'s list and <u>then</u> do $\underset{\sim}{\Lambda}$'s

$\underset{\sim}{\Omega}(\underset{\sim}{\Lambda}|V\rangle) = \underset{\sim}{\Omega}\underset{\sim}{\Lambda}|V\rangle$ means do $\underset{\sim}{\Lambda}$'s list and <u>then</u> do $\underset{\sim}{\Omega}$'s

<u>often</u> $\underset{\sim}{\Lambda}\underset{\sim}{\Omega}|V\rangle \neq \underset{\sim}{\Omega}\underset{\sim}{\Lambda}|V\rangle$

we say: $\underbrace{\underset{\sim}{\Omega}\underset{\sim}{\Lambda} - \underset{\sim}{\Lambda}\underset{\sim}{\Omega}}_{} \neq 0$

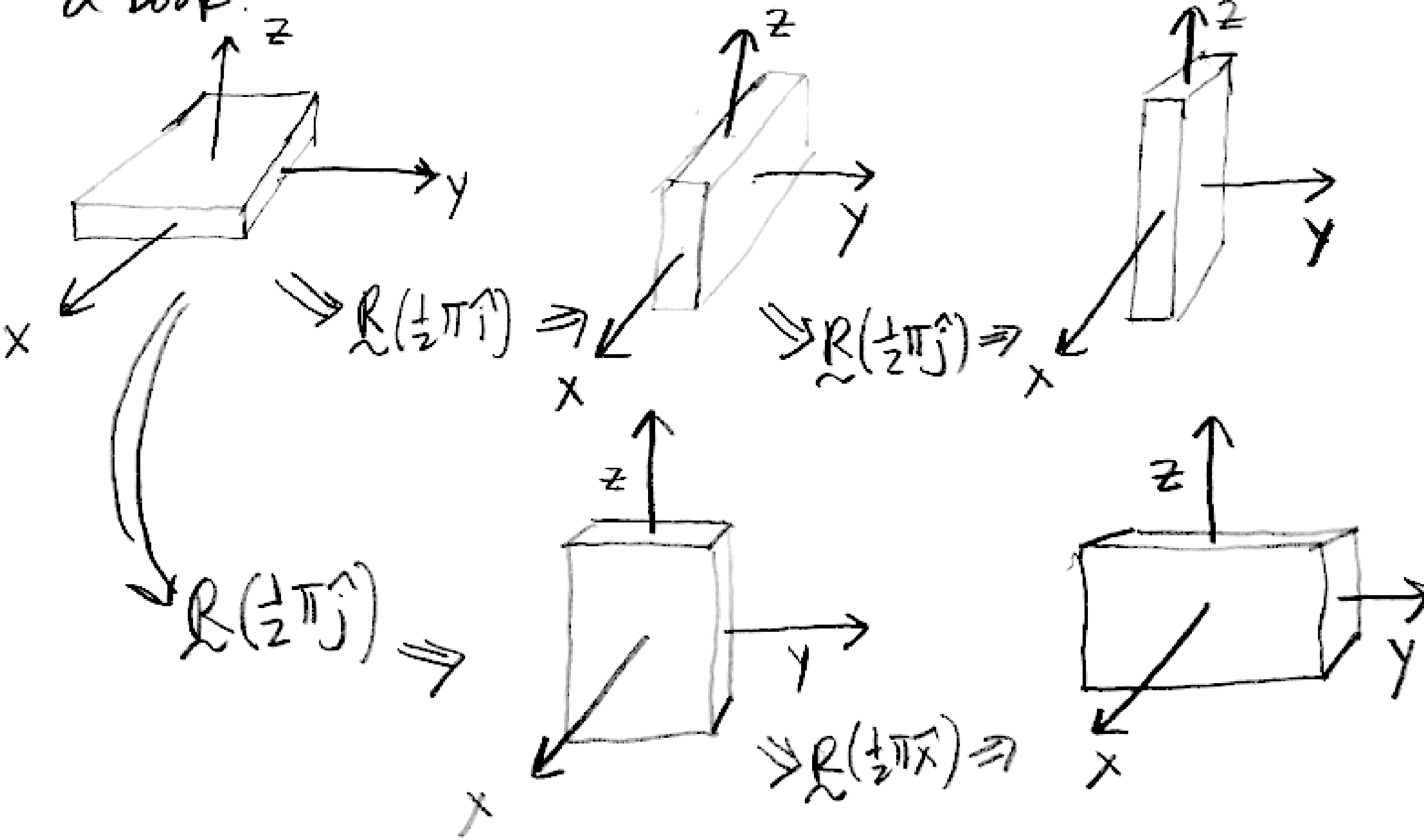
called the <u>commutator</u> of $\underset{\sim}{\Omega}$ and $\underset{\sim}{\Lambda}$

notation: $[\underset{\sim}{\Omega}, \underset{\sim}{\Lambda}] \equiv \underset{\sim}{\Omega}\underset{\sim}{\Lambda} - \underset{\sim}{\Lambda}\underset{\sim}{\Omega}$

<u>Example</u>: in 3-space, $\mathbb{V}^3_{xyz}(\mathbb{R})$

$$\underset{\sim}{R}\left(\tfrac{1}{2}\pi\hat{i}\right)\underset{\sim}{R}\left(\tfrac{1}{2}\pi\hat{j}\right) \neq \underset{\sim}{R}\left(\tfrac{1}{2}\pi\hat{j}\right)\underset{\sim}{R}\left(\tfrac{1}{2}\pi\hat{i}\right)$$

will do this as a demonstration on a book:



<u>Inverses</u> $\quad \underset{\sim}{\Omega}$ doesn't always have an inverse.

when $\underset{\sim}{\Omega}|V\rangle = 0$ and $|V\rangle \neq 0$, no inverse.

$$\underset{\sim}{\Omega}\,\underset{\sim}{\Omega}^{-1} = \underset{\sim}{\mathbb{1}} = \underset{\sim}{\Omega}^{-1}\underset{\sim}{\Omega}$$

$$(\underset{\sim}{\Omega}\underset{\sim}{\Lambda})^{-1} = \underset{\sim}{\Lambda}^{-1}\underset{\sim}{\Omega}^{-1} \qquad \underset{\sim}{\Omega}\underset{\sim}{\Lambda}\underbrace{(\underset{\sim}{\Lambda}^{-1}\underset{\sim}{\Omega}^{-1})}_{\underset{\sim}{\mathbb{1}}} = \underset{\sim}{\mathbb{1}}$$

order reversed

Matrix Elements

$$|V\rangle \leftrightarrow \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad \langle V| \leftrightarrow (v_1^* \; v_2^* \ldots v_n^*)$$

"project $|V\rangle$ onto an orthonormal basis"

that is $|V\rangle = \sum_{i=1}^{n} v_i |i\rangle \qquad \langle V| = \sum_{i=1}^{n} \langle i| v_i^*$

$= \sum_{i=1}^{n} |i\rangle\langle i|V\rangle \qquad = \sum_{i=1}^{n} \langle V|i\rangle\langle i|$

Linear Operators are <u>matrices</u>

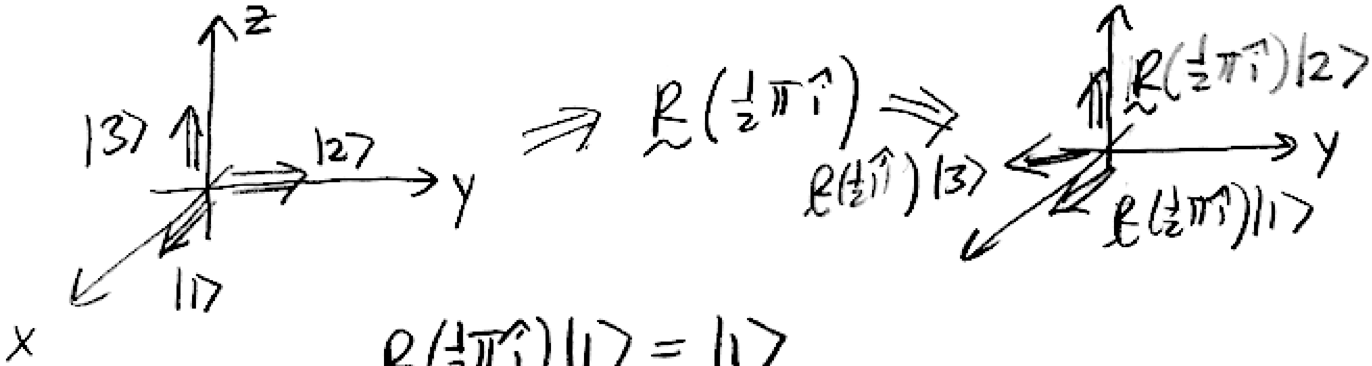
$$\underset{\sim}{\Omega} \leftrightarrow \begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \cdots & \Omega_{1n} \\ \Omega_{21} & & & & \\ \Omega_{31} & & & & \\ \vdots & & & & \\ \Omega_{n1} & & & & \Omega_{nn} \end{pmatrix}$$

"image or representation of $\underset{\sim}{\Omega}$"

The matrix representation $\boxed{\text{DEPENDS ON THE BASIS}}$. Which means, the same $\underset{\sim}{\Omega}$ can be reprented by two different matrices! Given the basis $|1\rangle, |2\rangle, \ldots, |n\rangle$

$$\Omega_{ij} = \langle i|\underset{\sim}{\Omega}|j\rangle, \neq \langle i'|\underset{\sim}{\Omega}|j'\rangle \; \left(\text{other basis}\right)$$

<u>Example</u>: $\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})$: use xyz basis



$$\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})|1\rangle = |1\rangle$$
$$\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})|2\rangle = |3\rangle$$
$$\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})|3\rangle = -|2\rangle$$

$\langle 1|\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})|1\rangle = \langle 1|1\rangle = 1$ $\quad \langle 1|\underset{\sim}{R}|2\rangle = \langle 1|3\rangle = 0$ $\quad \langle 1|\underset{\sim}{R}|3\rangle = -\langle 1|2\rangle = 0$

$\langle 2|\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})|1\rangle = \langle 2|1\rangle = 0$ $\quad \langle 2|\underset{\sim}{R}|2\rangle = \langle 2|3\rangle = 0$ $\quad \langle 2|\underset{\sim}{R}|3\rangle = -\langle 2|2\rangle = -1$

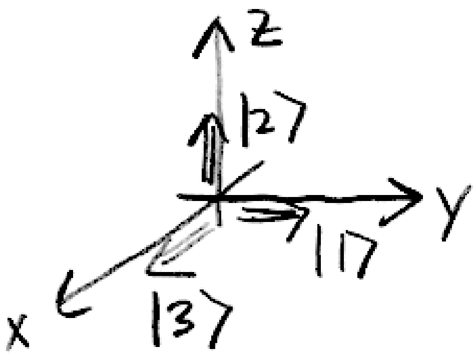
$\langle 3|\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i})|1\rangle = \langle 3|1\rangle = 0$ $\quad \langle 3|\underset{\sim}{R}|2\rangle = \langle 3|3\rangle = 1$ $\quad \langle 3|\underset{\sim}{R}|3\rangle = -\langle 3|2\rangle = 0$

$$\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i}) \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
or $\Longleftrightarrow$

symbol: $\doteq$ same as $\Longleftrightarrow$ "is represented by"

But I could have labelled....

in this basis....



$$\underset{\sim}{R}(\tfrac{1}{2}\pi\hat{i}) \doteq \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

worce, could have picked a basis not parallel to the x, y directions!