Vector Spaces with a Sense of Length

aka "inner product spaces"

Generalize the dot product, "scalar" product

1, 2, 3 -d space: $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$

$= |\vec{A}||\vec{B}|\cos\theta$

(1) dot product: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ (order unimportant).

inner product: for $|V\rangle, |W\rangle$ (two abstract vectors)

notation for inner product: $\langle V|W\rangle$

does $\langle V|W\rangle = \langle W|V\rangle$? $\boxed{\text{NOT ALWAYS}}$

if the "field" is real valued YES

if the "field" is complex valued NOT ALWAYS.

for a complex field ($a|V\rangle$ can have complex a)

$$\boxed{\langle V|W\rangle = \langle W|V\rangle^* \quad \text{complex conjugate.}}$$

that is, if $\langle V|W\rangle = 3+2i$ (for example)

$\langle W|V\rangle^* = 3+2i$

$\langle W|V\rangle^{**} = \langle W|V\rangle = (3+2i)^* = 3-2i$

(2) dot product: $\underbrace{\vec{A}\cdot\vec{A}}_{\text{real valued}} \geq 0$ $\left(\begin{matrix} A_x^2 + A_y^2 + A_z^2 \geq 0 \\ |\vec{A}|^2 > 0 \end{matrix} \right)$

inner product $\langle V|V\rangle$ = real valued ≥ 0

(3) linearity in $\boxed{\text{second}}$ abstract vector

$$\langle V|\left(a|W\rangle + b|Z\rangle\right) = a\langle V|W\rangle + b\langle V|Z\rangle$$

what about $\boxed{\text{first}}$ one? <u>CAREFUL</u>

notation: $a|W\rangle + b|Z\rangle$ call $|aW + bZ\rangle$

know $\langle V|aW+bZ\rangle = a\langle V|W\rangle + b\langle V|Z\rangle$

want $\langle aW+bZ|V\rangle = \langle V|aW+bZ\rangle^*$

$$= a^*\langle V|W\rangle^* + b^*\langle V|Z\rangle^*$$

$$\boxed{\langle aW+bZ|V\rangle = a^*\langle W|V\rangle + b^*\langle Z|V\rangle \quad !!}$$

ANTI-LINEARITY

note:

$$\langle W+Z|W+Z\rangle = \left(\langle W| + \langle Z|\right)\left(|W\rangle + |Z\rangle\right)$$

$$= \langle W|W\rangle + \langle Z|Z\rangle + \langle W|Z\rangle + \langle Z|W\rangle$$

$$= \underbrace{\langle W|W\rangle + \langle Z|Z\rangle}_{\text{Real, Positive}} + \underbrace{\langle W|Z\rangle + \langle W|Z\rangle^*}_{\substack{\text{Imaginary part cancels} \\ = 2\,\mathrm{Re}(\langle W|Z\rangle)}}$$

D8: $\langle v|w\rangle = 0$, say $|v\rangle, |w\rangle$ are orthogonal

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D9: $\sqrt{\langle v|v\rangle}$ is called the "norm" or length of $|v\rangle$

D10: A set of kets $|i\rangle$, $i = 1$ to n

with $\begin{cases} \langle i|i\rangle = 1 \\ \langle i|j\rangle = 0 \quad \text{when } i \neq j \end{cases}$

together, $\langle i|j\rangle = \delta_{ij}$ "Kronecker Delta"

is called an "orthonormal basis"

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then when $|v\rangle = \sum_i v_i |i\rangle$

$|w\rangle = \sum_i w_i |i\rangle$

$$\langle v|w\rangle = \sum_{i=1}^{n} v_i^* \langle i|w\rangle$$

antilinear linear

$$= \left(\sum_{i=1}^{n} v_i^* \langle i|\right)\left(\sum_{j=1}^{n} w_j |j\rangle\right)$$

$$= \sum_{i,j=1}^{n} v_i^* w_j \underbrace{\langle i|j\rangle}_{0 \text{ unless } i=j}$$

$$\langle v|w\rangle = \sum_{i=1}^{n} v_i^* w_i \,, \quad \langle w|v\rangle = \langle v|w\rangle^* = \sum_{i=1}^{n} v_i w_i^*$$

Visualization, Dual Space, Bras

$$|v\rangle = \sum_{i=1}^{n} v_i |i\rangle \quad \text{orthonormal basis.}$$

$$\langle j|v\rangle = \sum_{i=1}^{n} v_i \underbrace{\langle j|i\rangle}_{\delta_{ji}} = v_j \text{ alone}$$

$$|v\rangle = \sum_{i=1}^{n} \langle i|v\rangle |i\rangle = \underbrace{\sum_{i=1}^{n} |i\rangle\langle i|v\rangle}_{\text{"looks nice"}}$$

just # put on other side.

$$|v\rangle \leftrightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + v_n \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

(not unique).

$$|w\rangle \leftrightarrow \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$\langle v|w\rangle = \sum_{i=1}^{n} v_i^* w_i = \begin{bmatrix} v_1^* & v_2^* & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

suggests new relation:

$\langle v| \Rightarrow$ "bra-V", $\langle v| \leftrightarrow \begin{bmatrix} v_1^* & v_2^* & v_3^* & \cdots & v_n^* \end{bmatrix}$

Space of $|V\rangle$ is "Vector Space", $\rightarrow [\ ]$

Accessory Space of $\langle V| \rightarrow [^{*}\ ^{*}\ ]$

is called a "Dual Space".

Adjoint Operation

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}^{\dagger}$$

definition of $\dagger$

(a) Transpose the thing

(b) complex conjugate.

(a) $[V_1\ V_2\ V_3 \cdots\cdots V_n]$

(b) $[V_1^*\ V_2^* \cdots\cdots V_n]$

so if $|V\rangle \leftrightarrow \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$, $\langle V| \leftrightarrow [V_1^* V_2^* \cdots V_n^*]$

$|aV\rangle = a|V\rangle \leftrightarrow \begin{bmatrix} aV_1 \\ aV_2 \\ \vdots \\ aV_n \end{bmatrix}$ $\quad \langle aV| \leftrightarrow [a^*V_1^*, a^*V_2^* \cdots a^*V_n^*]$

inside + out same for kets.

inside

$\langle aV| = a^*\langle V|$

inside + out not same for bras.

more notation: orthonormal basis

$$|v\rangle = \sum_{i=1}^{n} v_i |i\rangle = \sum_{i=1}^{n} |i\rangle\langle i|v\rangle$$

$$\langle v| = \sum_{i=1}^{n} v_i^* \langle i| = \sum_{i=1}^{n} \langle i|v\rangle^* \langle i| = \sum_{i=1}^{n} \langle v|i\rangle\langle i|$$

compare with above

"To take the adjoint of an equation involving bras, kets, and coefficients, reverse the order of all the factors, exchange bras with kets, and complex conjugate the coefficients"

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an example interlude.

2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form a vector space.

what is the inner product?

→ guess that adjoint is involved.

$$|v\rangle \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\langle v| \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

but $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2+|c|^2 & a^*b+c^*d \\ ab^*+cd^* & |b|^2+|d|^2 \end{pmatrix}$

is a matrix, not a number!

Take the TRACE, the sum of the diagonal elements of a matrix

$$\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \mathrm{Tr}\begin{pmatrix} |a|^2+|c|^2 & a^*b+c^*d \\ ab^*+cd^* & |b|^2+|d|^2 \end{pmatrix}$$

$$= |a|^2+|b|^2+|c|^2+|d|^2 \geq 0$$

take the basis:

$$|1\rangle \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad |2\rangle \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad |3\rangle \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|4\rangle \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle 1|V\rangle = \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\dagger}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$= \mathrm{Tr}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = a$$

$$\langle 2|V\rangle = \mathrm{Tr}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\dagger}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$= \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = b \quad \left(\text{not obvious}\right).$$

an equally famous basis is:

$$\underset{\sim}{\sigma}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \underset{\sim}{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \underset{\sim}{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \underset{\sim}{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Gram-Schmidt

- Given n l.i. vectors, you can always derive an orthonormal basis (which is way more convenient to use).
- How? The Gram-Schmidt Procedure.

n l.i. vectors $\rightarrow$ call them $|I\rangle, |II\rangle, |III\rangle, \ldots$ (roman numerals).

want: $|1\rangle, |2\rangle, |3\rangle, \ldots \ldots |n\rangle$
with $\langle i|j\rangle = \delta_{ij}$; $\langle I|II\rangle \neq 0$ perhaps, etc.

1) set $|1\rangle = \dfrac{|I\rangle}{\sqrt{\langle I|I\rangle}}$ so $\langle 1|1\rangle = 1$

2) now $\langle 1|II\rangle = \dfrac{\langle I|II\rangle}{\sqrt{\langle I|I\rangle}} \neq 0$

but consider

$$|2'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle$$

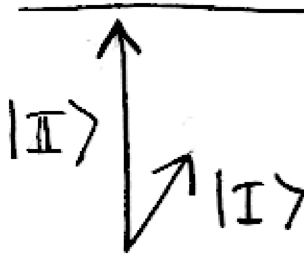
then $\langle 1|2'\rangle = \langle 1|II\rangle - \underbrace{\langle 1|1\rangle}_{1}\langle 1|II\rangle$

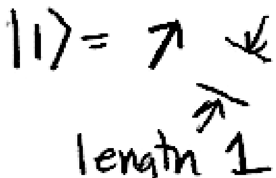
$= \langle 1|II\rangle - \langle 1|II\rangle = 0$

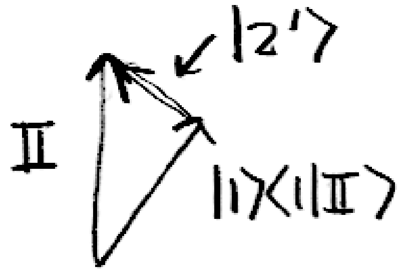
but $|2'\rangle$ may not be normalized,

so $$|2\rangle = \frac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}}$$ complicated in terms of $|1\rangle, |III\rangle, |I\rangle$.

<u>in 2 dimensions</u>:







$$|3'\rangle = |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle$$

$$|4'\rangle = |IV\rangle - |1\rangle\langle 1|IV\rangle - |2\rangle\langle 2|IV\rangle - |3\rangle\langle 3|IV\rangle$$

You get a different result for the basis if you shuffle the original n l.i. vectors.