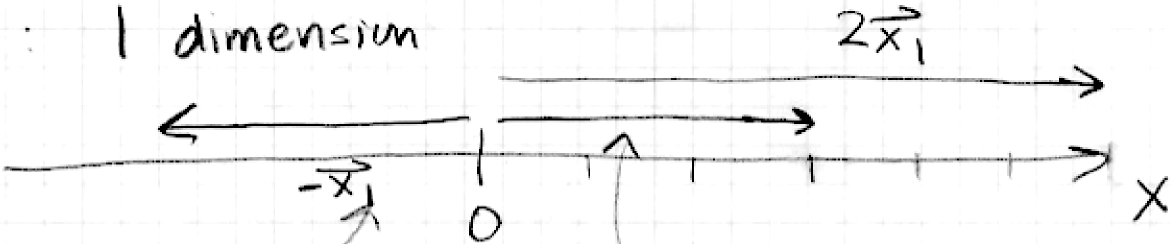
Mathematical Introduction (Shankar, Chap. 1)

→ Abstraction of the concept of the "vector"

"vector" → generally taught in lower division as a "magnitude" and a "direction"

very physical definition.

eg : 1 dimension



$\vec{x}_1 = 3$ in $+x$ direction

can multiply by, say 2 ; $2\vec{x}_1$ twice as long

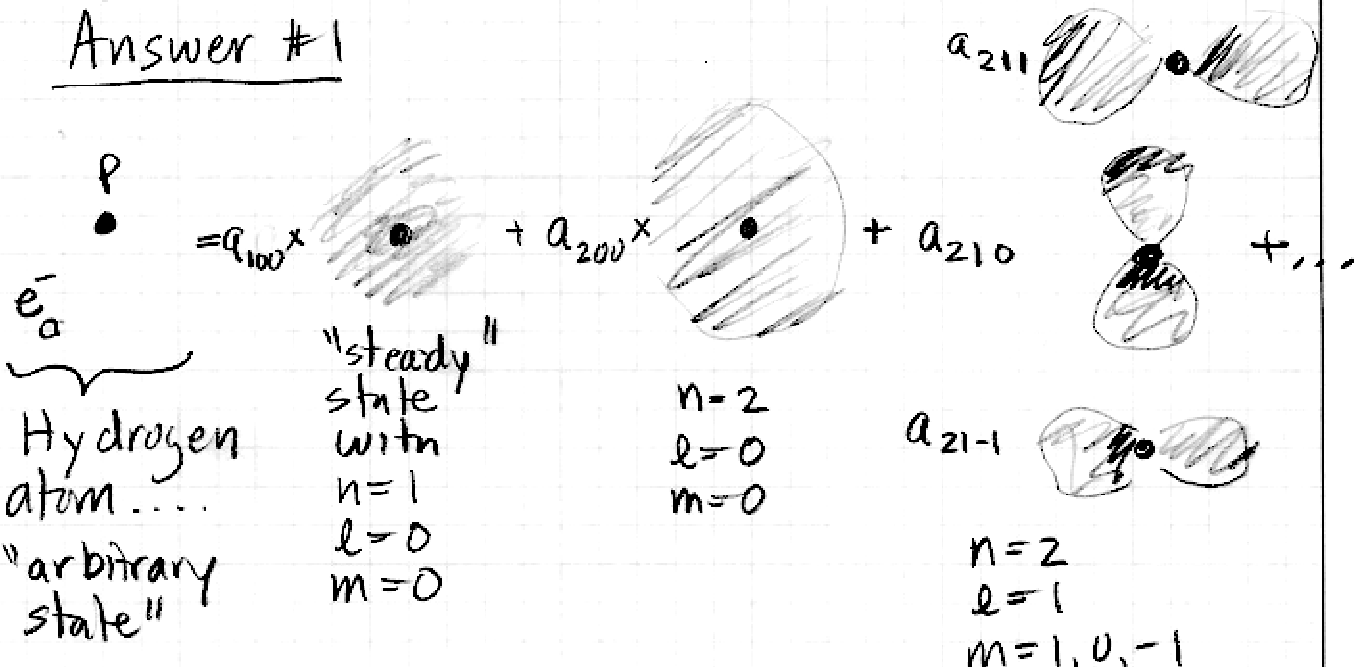
most interesting: multiplication by a negative number reverses direction

Mathematicians have "boiled" down the features that are important for defining the "linear vector space" and the "field" of numbers which can multiply the vectors

→ page 2 of Shankar

→ Why is this activity important for quantum mechanics?

Answer #1



$$\begin{bmatrix} a_{100} \\ a_{200} \\ a_{211} \\ a_{210} \\ a_{21-1} \\ \vdots \end{bmatrix} \leftarrow$$

- vector with many components
- meaning of number is "amount" of a given state in the "arbitrary state"

Answer #2 : SPIN

spin-$\frac{1}{2}$: $\nearrow\!\!\!\nearrow = a \times \Uparrow + b \times \Downarrow$ 2-d

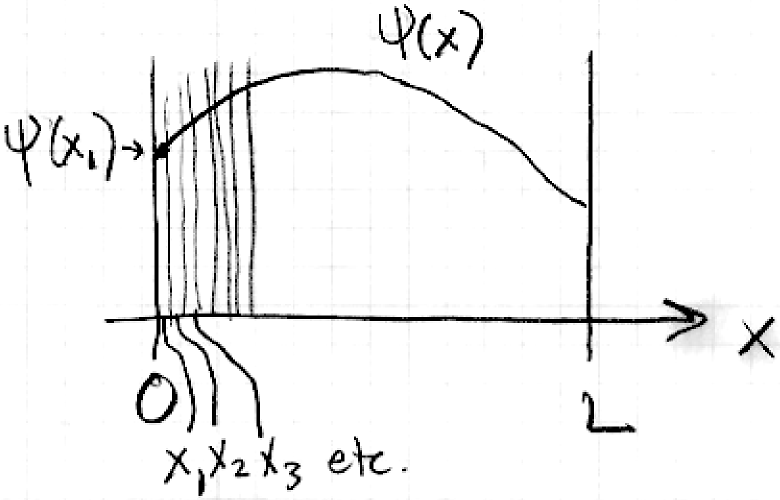
spin-up spin-down

but... a + b must be complex numbers.

spin-1 : $\nearrow\!\!\!\nearrow = a\Uparrow + b\Rightarrow + c\Downarrow$ 3-d

-$\frac{3}{2}$ $\nearrow\!\!\!\nearrow = a\Uparrow + b\uparrow + c\downarrow + d\Downarrow$ 4-d

Answer # 3 : Wave functions



$$\begin{bmatrix} \psi(x_1) \\ \psi(x_2) \\ \vdots \\ \psi(x_i) \\ \vdots \\ i \to \infty ! \end{bmatrix}$$

$\Leftarrow$ vector with an infinite number of dimensions

Interesting concept $\leftrightarrow$ "dimensionality" of the vector space.

Notation: not $\vec{V}$ (implies 2-d or 3-d)

but $|v\rangle$ say "ket v"

- could have ∞ dimensions
- satisfies rules on page 2

$|v\rangle + |-v\rangle = 0$; $|-v\rangle = (-1)\cdot|v\rangle$

inverse under addition

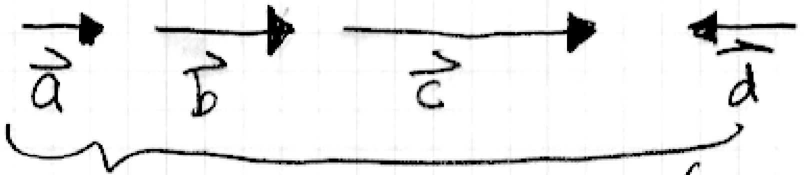
$|v\rangle$ is "abstract" but can be "imaged" or "represented" by organizing its components in a vector, matrix, or function

depending upon dimensionality, might have

$$|v\rangle \leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \quad \underset{\sim}{\text{or}} \quad |v\rangle \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \underset{\sim}{\text{or}} \quad |v\rangle \leftrightarrow \psi(x)$$

to deal with dimensionality, first need concept of linear dependence + independence

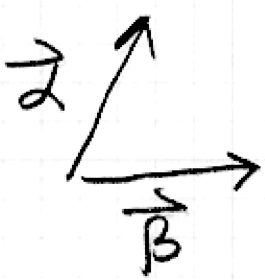
Concept: think 2 dimensions



all linearly dependent (parallel)

why? $\vec{b} = (\text{number}) \times \vec{a}$ et. cetera

but



$\vec{\alpha}$ and $\vec{\beta}$ are linearly independent because they are not parallel.

NEVER Have $\vec{\beta} = (\text{number}) \times \vec{\alpha}$

never have $\vec{\beta} - (\text{number}) \times \vec{\alpha} = 0$

never have $a_1 \vec{\beta} + a_2 \vec{\alpha} = 0$

Definition: given n vectors

#3 in book p. 4

$|1\rangle, |2\rangle, |3\rangle, \ldots, |n\rangle$

these are linearly independent if the only way

$$\sum_{i=1}^{n} a_i |i\rangle = 0 \quad \text{is} \quad a_1 = a_2 = \ldots a_n = 0$$

$\uparrow$ real numbers

say

$$|1\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$$

are these linearly independent?

note:
$$-2|2\rangle = \begin{pmatrix} -2 & -2 \\ 0 & -2 \end{pmatrix}$$
$$|1\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$-2|2\rangle + |1\rangle = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$$

so $|1\rangle - 2|2\rangle - |3\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

not linearly independent (Exercise 1.1.4).

Definition: A vector space has dimension n if it can accomodate a maximum of n linearly independent vectors.

$\mathbb{V}^n(R) \leftarrow$ n dimensional, #'s are real

#4 in book p.5

$\mathbb{V}^n(C) \leftarrow$ n dimensional, #'s are complex.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 4 dimensional (even if a, b, c, d complex).

pretty clear

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

the 4 l.i. vectors

$|1\rangle \quad |2\rangle \quad |3\rangle \quad |4\rangle$

Thm #1: any vector $|V\rangle$ in an n-dim. space can be written as a linear combination of n linearly independent vectors $|1\rangle \ldots |n\rangle$.

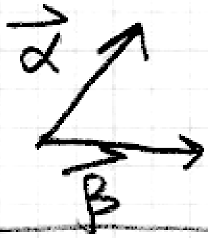
Proof: if $|V\rangle \neq \sum_{i=1}^{n} v_i |i\rangle$ for any set of v_i

then $|V\rangle - \sum_{i=1}^{n} v_i |i\rangle \neq 0$ "

and there are $n+1$ linearly independent vectors... this contradicts the fact that the dimensionality was n.

Definition: A set of n linearly independent vectors in an n-dimensional space is called a basis (#5 in book, p. 6

comment: NEED NOT BE ORTHOGONAL!



$\leftarrow$ in 2-d, these are a basis!

Definition: The coefficients of expansion v_i of a vector in terms of a basis ($|i\rangle$) are called the components of the vector in that basis

$$(*) \quad |V\rangle = \sum_{i=1}^{n} v_i |i\rangle$$

could imagine as

"imaging" or "representing" $|V\rangle$

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix}$$

Theorem: (*) is unique.

Proof: $|v\rangle = \sum_{i=1}^{n} v_i |i\rangle = \sum_{i=1}^{n} v_i' |i\rangle$

then $\sum_{i=1}^{n} (v_i - v_i')|i\rangle = 0$

basis!
linearly independent!

$\therefore$ all $v_i - v_i' = 0$

$v_i = v_i'$

also: if $|v\rangle = \sum_{i=1}^{n} v_i |i\rangle$

$|w\rangle = \sum_{i=1}^{n} w_i |i\rangle$

components are sum of others

then $|v\rangle + |w\rangle = \sum_{i=1}^{n} (v_i + w_i)|i\rangle$

(can use page 2 text to go through)

also $a|v\rangle = \sum_{i=1}^{n} a v_i |i\rangle$

components are multiples of old components